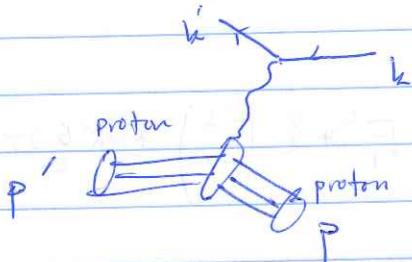


18 March, Tuesday

Deep Inelastic Scattering and Parton Model

We would like to understand structure of the proton (nucleon) and nature of strong interactions.

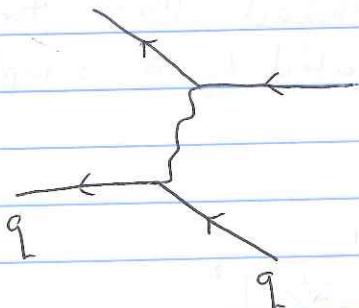
Through ep scattering we learned proton does not behave like a point object. For elastic scattering



$$\frac{d\sigma}{dQ^2} \sim \frac{d\sigma}{dQ^2} \Big|_{M^2_W} \cdot |F(-q^2)|^2$$

$$F(-q^2) \sim \frac{1}{[1 - \frac{q^2}{m_e^2}]^2}$$

We can actually predict the asymptotic q^2 behavior w/ simple quark counting rules.



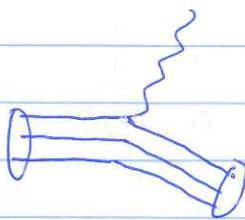
$$\langle q(p') | f^\mu | q(p) \rangle$$

but $\vec{p} = 0$

$$\sim \bar{u}(q) \gamma^\mu u(0) \sim Q$$

$$(Q^2 = -q^2)$$

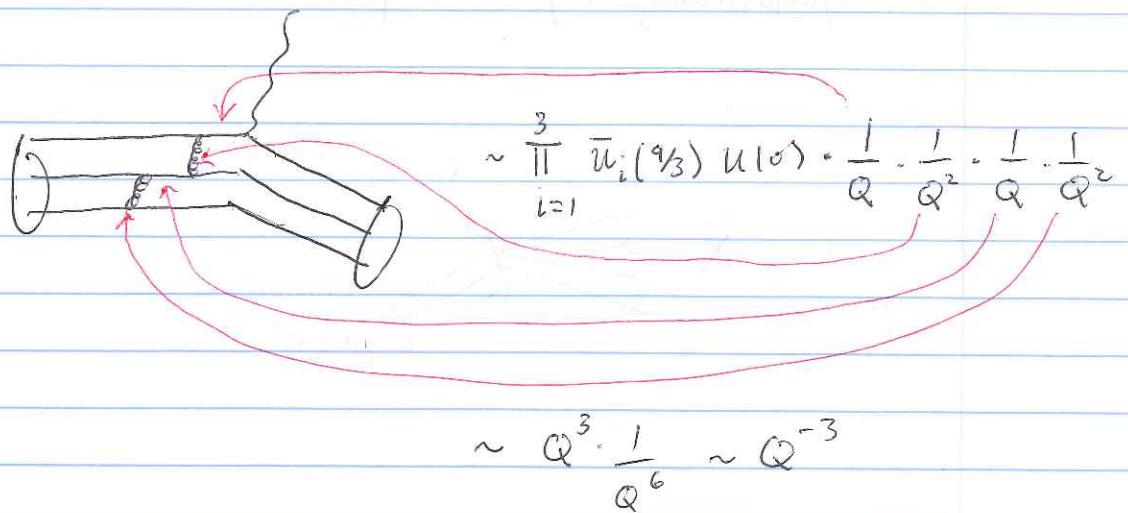
What if we embed the quark in a proton?



If all 3 quarks start "at rest" and one of them is kicked hard by the virtual photon, what is the probability the proton will stay together?

$$\lim_{q \rightarrow 0} \langle q(q) | q(0) \rangle \sim 0$$

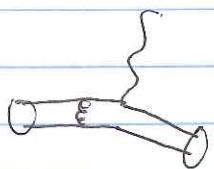
But this quark can share the large momentum w/other quarks through gluon exchange



If we compare to elastic scattering off quark

$$\sim \text{---} \cdot \frac{1}{Q^4}$$

So what would we predict for pions?



$$\sim (\bar{u}(q_i)u)^2 \cdot \frac{1}{Q} \cdot \frac{1}{Q^2}$$

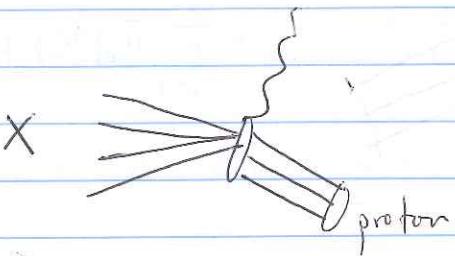
$$\sim Q \cdot \frac{1}{Q^2}$$

and indeed, the ~~proto~~ pion elastic form factor scales asymptotically like $1/Q^2$

Now, what about inelastic proton scattering?

At low energies, there is a lot of messy resonance structure. At higher energies, the cross section smoothes out.

What we would like to understand, is behavior at asymptotically high energies



As we are interested in understanding general structure of the proton, we can consider inclusive scattering

$$\sum_X e(k-q) + X(p+q) \leftarrow e(k) + p(p)$$

$$(p+q)^2 \gg m_p^2 \quad \text{Deep inelastic scattering}$$

This is experimentally "clean": just detect final electron and its momentum.

This is also theoretically clean. We can use the optical theorem to relate the total inclusive cross section to the forward Compton scattering amplitude

$$\left| \times \begin{array}{c} q \\ \diagup \quad \diagdown \\ k \end{array} p \right|^2 \sim \begin{array}{c} \text{wavy lines} \\ \diagup \quad \diagdown \\ \text{wavy lines} \end{array}$$

This denotes we are

taking the Imaginary part
along the 'cut'

The inclusive cross section is

$$\frac{d\sigma}{dQ^2 dE'} = \frac{\alpha^2}{Q^4} \left(\frac{E'}{E} \right) \ell^{\mu\nu} W_{\mu\nu}$$

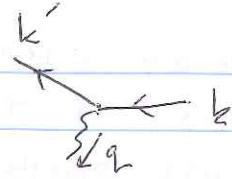
↑ leptonic tensor ↗ hadronic tensor

If we consider ν scattering also

$$\frac{1}{Q^4} \rightarrow \left(\frac{1}{Q^2 + M^2} \right)^2, \quad \mu = (M_W, M_Z)$$

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The leptonic tensor is



$$\ell^{\mu\nu} = \frac{1}{2} \text{tr} (k' \gamma^\mu k \gamma^\nu)$$

sum over final spins
 average over initial spins

$$= \frac{1}{2} \cdot (4k^\mu k^\nu + 4k^\mu k^\nu + 2q^2 g^{\mu\nu})$$

$$\text{tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g_{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

1st & 3rd term give $k k' + k' k$

$$\begin{aligned} k \cdot k' g^{\mu\nu} &= (k - q) \cdot k' g^{\mu\nu} \\ &= (k^2 - q \cdot k) g^{\mu\nu} \\ &= (m_e^2 - q \cdot k) g^{\mu\nu} \\ &\approx -q \cdot k g^{\mu\nu} \end{aligned}$$

$$\begin{aligned} &\approx \frac{1}{2} ((k - q)^2 - k^2 - q^2) g^{\mu\nu} \\ &\approx \frac{1}{2} [m_e^2 - m_e^2 - q^2] g^{\mu\nu} \\ &\approx -\frac{1}{2} q^2 g^{\mu\nu} \end{aligned}$$

What about hadronic tensor?

$$W_{\mu\nu} = \frac{1}{4M} \sum_\sigma \int d^4x \langle p, \sigma | J_\mu(0) | \bar{x} x x | J_\nu(0) | p, \sigma \rangle (2\pi)^3 \delta^4(p_x - p - q_x)$$

$$= \frac{1}{8\pi M} \sum_\sigma \int d^4x e^{iq \cdot x} \langle p, \sigma | J_\mu(x) J_\nu(0) | p, \sigma \rangle$$

How can we see this?

$$W_{\mu\nu} = \frac{1}{8\pi M} \sum_{\sigma} \int d^4 z e^{i q \cdot z} \langle p\sigma | J_\mu(z) J_\nu(0) | p\sigma \rangle$$

i) insert complete set of states

$$1 = \sum_x |x\rangle \langle x|$$

$$\sum_x = \sum \int \frac{d^3 p_x}{(2\pi)^3} \frac{1}{2E_x}$$

$$= \frac{1}{8\pi M} \sum_{\sigma} \sum_x \int d^4 z e^{i q \cdot z} \langle p\sigma | J_\mu(z) | x X x \rangle \langle x X x | J_\nu(0) | p\sigma \rangle$$

2) use translation invariance to shift $J_\mu(z)$

$$= \frac{1}{8\pi M} \sum_{\sigma} \sum_x \int d^4 z e^{i q \cdot z} \langle p\sigma | e^{i \hat{p} \cdot z} J_\mu(0) e^{-i \hat{p} \cdot z} | x X x \rangle \langle x X x | J_\nu(0) | p\sigma \rangle$$

3) notice only z dependence now in exponentials

$$e^{i z \cdot (q + p - \hat{p}_x)}$$

so we can perform $\int d^4 z$

$$= \frac{1}{8\pi M} \sum_{\sigma} \sum_x (2\pi)^4 \delta^4(p_x - q - p) \langle p\sigma | J_\mu(0) | x X x \rangle \langle x X x | J_\nu(0) | p\sigma \rangle$$

$$= \frac{1}{4M} \sum_{\sigma} \sum_x (2\pi)^3 \delta^3(p_x - q - p) \langle p\sigma | J_\mu | x X x \rangle \langle x X x | J_\nu | p\sigma \rangle$$

It is sometimes useful to write the tensor w/ commutator of currents. To show we can do this, consider

$$\begin{aligned}
 & \int d^4z e^{iq\cdot z} \langle p\sigma | J_{\nu(0)} J_{\mu}(z) | p\sigma \rangle \\
 &= \oint_x \int d^4z e^{iq\cdot z} \langle p\sigma | J_{\nu(0)} | \times X \times | e^{ip\cdot z} \hat{J}_{\mu}(z) e^{-ip\cdot z} | p\sigma \rangle \\
 &= \oint_x \int d^4z e^{iz \cdot (q-p+P_x)} \langle p\sigma | J_{\nu} | \times X \times | \hat{J}_{\mu} | p\sigma \rangle \\
 &= \oint_x (2\pi)^4 \delta^4(P_x - (p-q)) \langle p\sigma | J_{\nu} | \times X \times | \hat{J}_{\mu} | p\sigma \rangle
 \end{aligned}$$

Notice the $\delta^0(E_x - (M_N - q^0))$:

for the nucleon, there is no possible intermediate state with $E < M_N$, so this entire contribution is \emptyset

$$\Rightarrow W_{\mu\nu} = \frac{1}{8\pi M} \sum_{\sigma} \int d^4z e^{iq\cdot z} \langle p\sigma | [J_{\mu}(z), J_{\nu}(z)] | p\sigma \rangle$$

notice this hadronic tensor depends upon 2 kinematic variables

$$W_{\mu\nu} = W_{\mu\nu}(p, q)$$

What is the most general form of $W_{\mu\nu}$?

$$W_{\mu\nu} = -W_1 g_{\mu\nu} + W_2 \frac{P_\mu P_\nu}{M^2} - i W_3 \epsilon_{\mu\nu\rho\sigma} \frac{P^\rho q^\sigma}{M^2} \\ + W_4 \frac{q^\mu q_\nu}{M^2} + W_5 \frac{P_\mu q_\nu + P_\nu q_\mu}{M^2} + i W_6 \frac{P_\mu q_\nu - P_\nu q_\mu}{M^2}$$

Recall the leptonic tensor

$$\ell^{\mu\nu} = 2 k^\mu k'^\nu + 2 k^\nu k'^\mu + q^2 g^{\mu\nu}$$

so with lepton scattering, no sensitivity to W_6

If we ignore ν -scattering for now, no sensitivity to W_3

Current conservation

$$q^\mu W_{\mu\nu} = 0$$

$$q^\nu W_{\mu\nu} = 0$$

$$q^\mu W_{\mu\nu} = -W_1 q_\nu + W_2 \frac{q \cdot P}{M^2} P_\nu + W_4 \frac{q^2 q_\nu}{M^2} + W_5 \left(\frac{q \cdot P q_\nu + q^2 P_\nu}{M^2} \right) = 0$$

$$= q_\nu \left[-W_1 + \frac{q^2}{M^2} W_4 + \frac{q \cdot P}{M^2} W_5 \right] = 0$$

$$+ P_\nu \left[\frac{q \cdot P}{M^2} W_2 + \frac{q^2}{M^2} W_5 \right]$$

$$W_5 = -\frac{q \cdot P}{q^2} W_2$$

$$W_4 = \frac{M^2}{q^2} W_1 + \frac{(P \cdot q)^2}{q^4} W_2$$

For ep scattering, The most general hadronic tensor

$$W_{\mu\nu} = W_1(p, q) \left(-g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right)$$

$$+ W_2(p, q) \frac{1}{M^2} \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right)$$

We can express the dependence upon p, q in terms of two Lorentz scalars

$$q^2 = (k' - k)^2$$

$$p \cdot q = M v \quad \text{in proton rest frame}$$

$$p \equiv (M, \vec{0})$$

$$p \cdot q = M q^0 \\ \equiv M v$$

v = virtual photon energy

There are also two convenient and commonly used dimensionless variables used to describe the kinematics

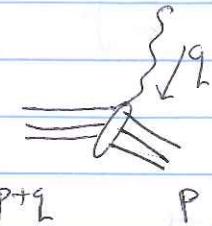
$$x = \frac{-q^2}{2Mv} \quad y = \frac{p \cdot q}{p \cdot k} \Rightarrow \frac{E' - E}{E} \quad \left. \begin{array}{l} \text{seen from} \\ \text{proton rest frame} \end{array} \right\}$$

We are interested in 'space-like' momentum transfers

$$Q^2 = -q^2$$

What are the bounds of these variables?

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1$$



$$(p+q)^2 = w^2 \equiv \text{Invariant mass of state } X$$

$$\begin{aligned} (p+q)^2 &= p^2 + 2p \cdot q + q^2 \\ &= M^2 + 2Mv - Q^2 \\ &= M^2 + 2Mv \left(1 - \frac{Q^2}{2Mv}\right) \\ &= M^2 + 2Mv(1 - x) \end{aligned}$$

$$\text{so } x = 1 = \text{elastic scattering } (p+q)^2 = M_p^2$$

$$y = \frac{p \cdot q}{p \cdot k} = \frac{p_0(k' - k)}{p_0 k} \quad \begin{array}{l} \text{forward scattering } k' = k \\ y = 0 \end{array}$$

Contracting w/ the lepton tensor, we arrive at
the cross section

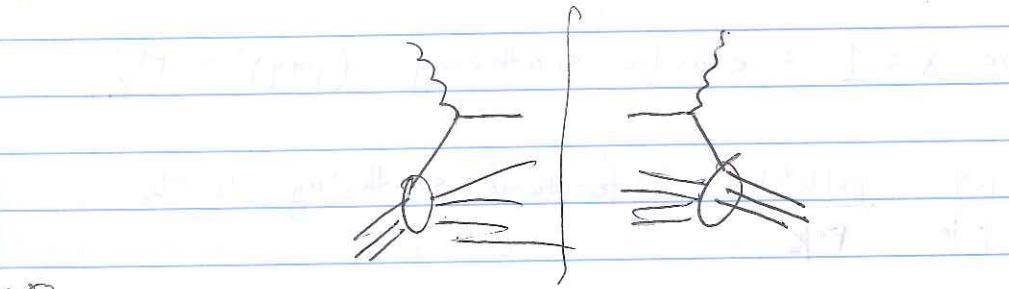
$$\frac{d\sigma}{d\Omega dE} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \left(2W_1 \sin^2 \frac{\theta}{2} + W_2 \cos^2 \frac{\theta}{2} \right)$$

By varying kinematics, we can measure

$$W_i(v, q^2)$$

What can we learn / deduce from DIS?

Imagine we have a highly virtual space like photon. This photon will have the resolving power to see "individual quarks"



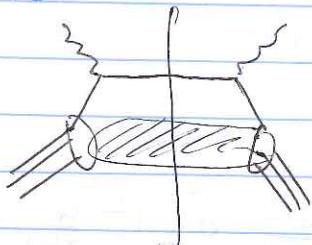
~~W_{1,2}~~

$$W_{\mu\nu} = \frac{1}{8\pi M^2} \int d^4 z e^{iq\cdot z} \langle \bar{q}q | J_\mu(z) J_\nu(0) | p\bar{p} \rangle$$

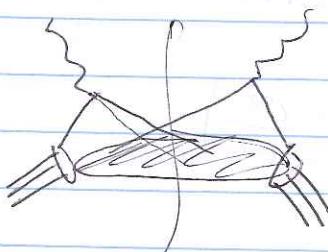
$$J_\mu = \sum_f e_f \bar{\psi}_f \gamma_\mu \psi_f$$

$$W_{\mu\nu} = \frac{1}{8\pi M} \sum_{f,f'}^2 \int d^4x e^{i\vec{q}\cdot\vec{x}} \langle p\sigma | \bar{\psi}_f(x) \gamma_\mu \psi_f(x) \bar{\psi}_{f'}(0) \gamma_\nu \psi_{f'}(0) | p\sigma \rangle$$

either there is a quark self contraction, or
the quarks return to the hadron in intermediate
state



handbag diagram



cats ears

We can qualitatively argue (and you can rigorously show)
the "cats ears" type of process is Q^2 suppressed
compared to the hand bag diagram

There are more quarks with large Q^2 flowing
through them

Suppose we postulate ~~at high Q^2~~ for DIS, we
can imagine $W_{\mu\nu}$ is given by sum over VCS
off individual quarks

$$\frac{d\bar{\sigma}}{dQ^2} = \frac{\alpha^2 e_f^2}{4k^2 \sin^4 \frac{\theta}{2}} \frac{\pi}{k' k} \left[\cos^2 \frac{\theta}{2} + \frac{Q^2}{2M^2} \sin^2 \frac{\theta}{2} \right] \quad \text{for point scattering of quark of flavor } f$$

$$dQ^2 = 2k'^2 d\cos \theta, \quad k' = E'$$

Elastic scattering $Q^2 = 2Mv$

$$\frac{d\bar{\sigma}}{dQ^2 dv} = \frac{\alpha^2}{4k^2 \sin^4 \frac{\theta}{2}} \frac{\pi}{k^2} \left[\cos^2 \frac{\theta}{2} + \frac{Q^2}{2M^2} \sin^2 \frac{\theta}{2} \right] \delta(v - \frac{Q^2}{2M})$$

$$\Rightarrow W_1 \approx \frac{Q^2}{6M^2}$$

$$W_1(v, Q^2) = \frac{Q^2}{4M^2} \delta(v - \frac{Q^2}{2M}) = \frac{Q^2}{2M^2 v} \delta(1 - \frac{Q^2}{2Mv})$$

$$W_2(v, Q^2) = \delta(v - \frac{Q^2}{2M}) = \frac{1}{v} \delta(1 - \frac{Q^2}{2Mv})$$

If the proton is a bound state of these quarks,
then each quark has charge e_f

carries momentum $x_f p$

has "mass" $M \rightarrow x_f M$

$$\Rightarrow W_1^f = \frac{e_f^2 Q^2}{4x_f M^2 v} \delta(x_f - \frac{Q^2}{2Mv})$$

$$W_2^f = \frac{e_f^2 x_f}{v} \delta(x_f - \frac{Q^2}{2Mv})$$

It is common to rescale these structure functions

$$F_1^f(x, Q^2) = M W_1^f(v, Q^2) = \frac{1}{2} e_f^2 S\left(x_f - \frac{Q^2}{2Mv}\right)$$

$$F_2^f(x, Q^2) = v W_2^f(v, Q^2) = e_f^2 x_f S\left(x_f - \frac{Q^2}{2Mv}\right)$$

notice the relation



We then need to understand how these quarks are distributed inside the proton wave function.

\Rightarrow parton distribution function (PDF)

$q_f(x_f) \equiv$ the probability density of finding quark-f in the proton with momentum fraction x_f

$$\Rightarrow F_1^f(x, Q^2) = \sum_f \int_0^1 dx_f q_f(x_f) \frac{e_f^2}{2} S\left(x_f - \frac{Q^2}{2Mv}\right)$$

$$\frac{Q^2}{2Mv} = x$$

$$= \sum_f q_f(x) \frac{e_f^2}{2}$$

similarly

$$F_2^f(x, Q^2) = \sum_f e_f^2 x_f q_f(x)$$

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Note the relation

$$2 \times F_1(x, Q^2) = F_2(x, Q^2)$$

Callan-Gross relation. (1969)

This was one of the crucial pieces of information that led us to believe the partons were fundamental spin- $\frac{1}{2}$ constituents

Turns out, you can measure longitudinal and transverse scattering off proton and find

$$R \equiv \frac{\sigma_L}{\sigma_T} \sim \frac{F_2 - 2 \times F_1}{2 \times F_1}$$

if partons were spin-0, then $F_1 = 0$
 $F_2 \neq 0$

but we have observed that experimentally, $F_2 \sim 2 \times F_1$

Also NOTE:

$$F_1(x, Q^2) = F_1(x) = \frac{1}{2} \sum_f e_f^2 q_f(x)$$

$$F_2(x, Q^2) = F_2(x) = \sum_f e_f^2 x q_f(x)$$

These structure functions are independent of the scale Q^2 .

Bjorken first noticed these were approximately true in the SLAC data.

The scaling led to the invention of the parton model (Feynman)

What were the crucial steps?

in DIS Limit $\left\{ \begin{array}{l} Q^2 \rightarrow \infty \\ v \rightarrow \infty \end{array} \right. \quad x = \frac{Q^2}{2Mv} = \text{fixed} \right\}$

the fundamental constituents of the proton must be weakly interacting spin- $\frac{1}{2}$ fermions

What also makes this useful is the PDFs are universal functions of a hadron (independent of process)